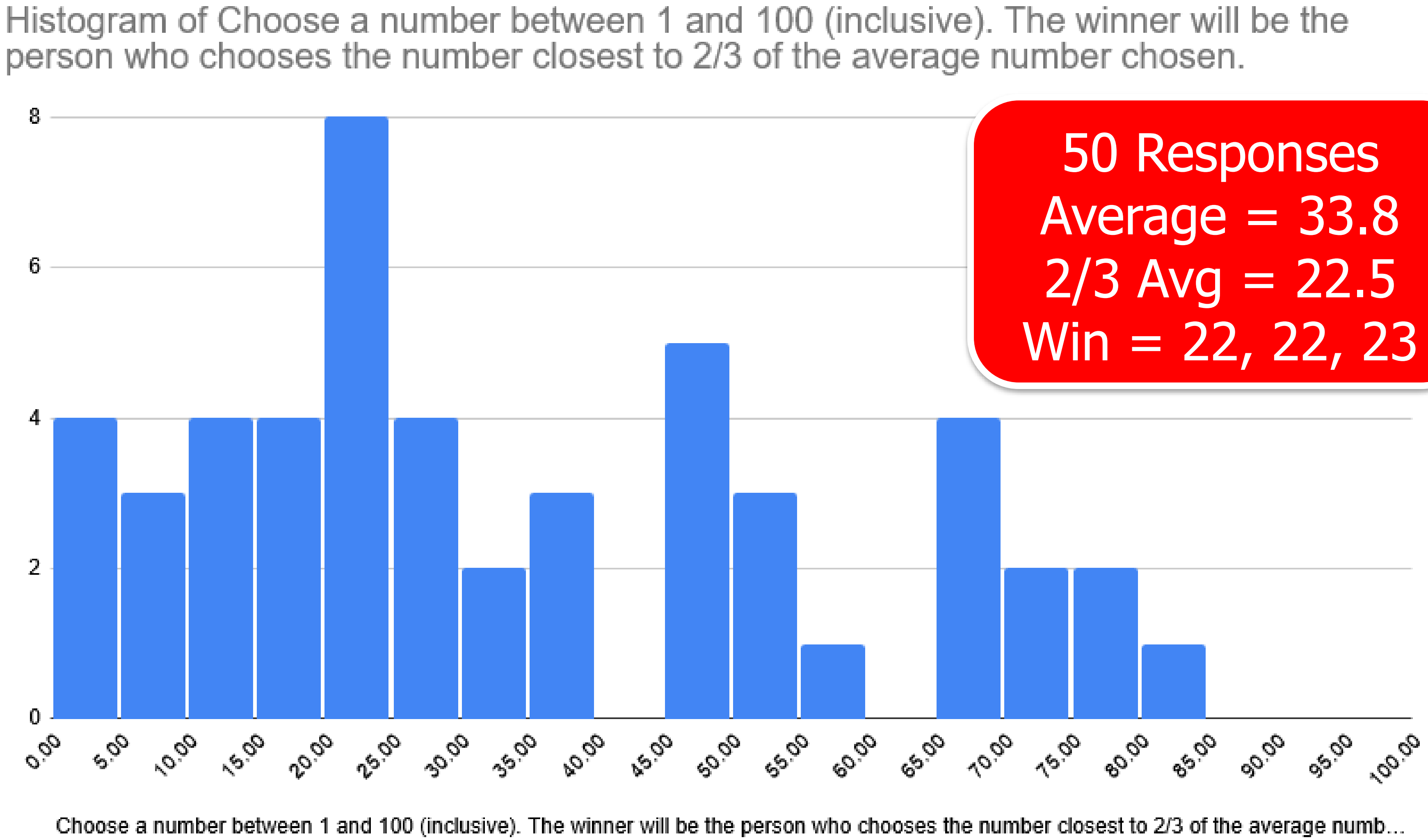
COMP 3200
Artificial Intelligence

Lecture 8

Intro to Game Theory
Matrix Games
Nash Equilibrium

Game: Choosing Numbers

- Everyone in class secretly choose a number between 1 and 100
- We will calculate the average number chosen for everyone in class
- The winner will be the person closest to 2/3 the average number chosen



50 Responses
Average = 33.8
2/3 Avg = 22.5
Win = 22, 22, 23

Game Theory

- Mathematical models for decision making
- Analysis of strategic actions in scenarios which can be defined as games
- Many uses in social sciences, computer science, economics, everyday life
- Nobody actually teaches us how to make decisions, so let’s look at the math

Game 1: Grades Game

- Two players secretly and simultaneously make a choice, with a defined outcome
  - Choices: Choose alpha (α) or beta (β)
- You choose α they choose α: B-
- You choose α they choose β: A
- You choose β they choose α : C
- You choose β they choose β: B+

Game 1: Grades Game

Partner

Me		α	β
	α	B-	A
	β	C	B+

My Grades

Game 1: Grades Game

		Partner	
		α	β
Me	α	B-	A
	β	C	B+

My Grades

		Partner	
		α	β
Me	α	B-	C
	β	A	B+

Partner Grades

Grades Game

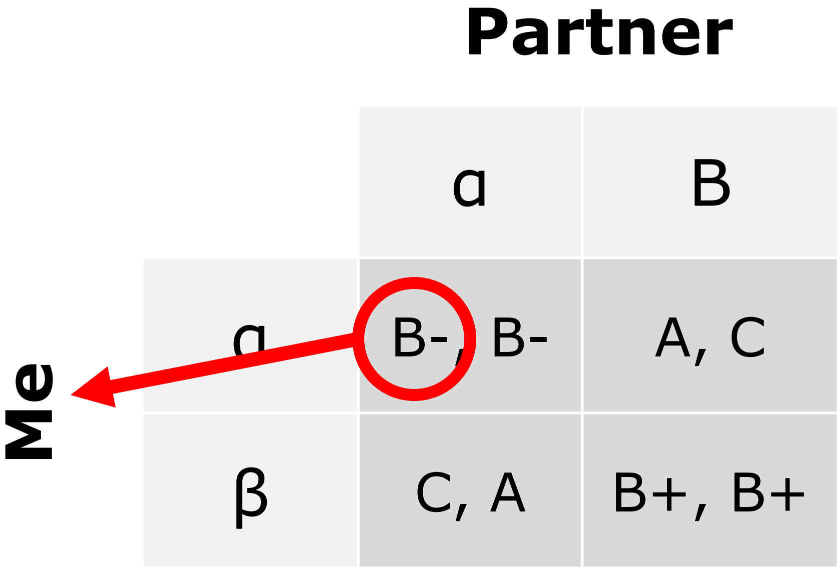
Partner

Me

	α	B
α	B-, B-	A, C
β	C, A	B+, B+

Outcome Matrix

Grades Game



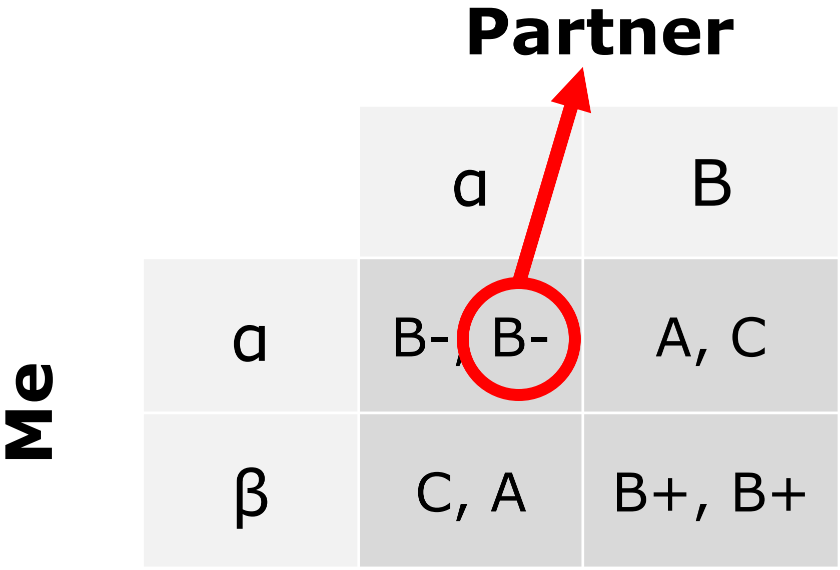
Partner

Me

	ɑ	B
ɑ	B-, B-	A, C
β	C, A	B+, B+

Outcome Matrix

Grades Game



		Partner	
		α	B
Me	α	B-, B-	A, C
	β	C, A	B+, B+

Outcome Matrix

Grades Game

Partner

Me

	α	B
α	B-, B-	A, C
β	C, A	B+, B+

Outcome Matrix

Grades Game

		Partner	
		α	B
Me	α	B-, B-	A, C
	β	C, A	B+, B+

Outcome Matrix

Grades Game

- We have actions, strategies, outcomes
- Not yet a game, what are we missing?
- Payoffs
  - What is our objective or goal in the game?
  - Game theory can’t help decide your objectives / payoffs
  - If you have payoffs, GT can help you achieve them
- Examples:
  - We care about our own grade
  - We care about others’ grades

Payoff Examples

- Payoff = assign numerical value to outcome
  - How much do I like the outcome?
  - The more I desire it, the higher the payoff
- Example Payoff
  - Imagine one of the players is greedy and assigns higher payoffs if they do better than partner
  - If both get B-, payoff is neural (0)
  - If I get A and you get C, payoff is good (3)

Grades Game Payoffs

Partner

Me

	α	B
α	B-, B-	A, C
β	C, A	B+, B+

Grade Results

Partner

Me

	α	β
α	0, 0	3, -1
β	-1, 3	1, 1

Payoff Matrix

Grades Game Payoffs

Partner

		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Payoff Matrix

- Numbers = Utility
- Player should attempt to maximize utility
- This example
  - (B-,B-) -> (0)
  - (A, C) -> (3)
  - Player cares about self

Grades Game Payoffs

		Partner	
		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Payoff Matrix

- What should you do?
- Consider all choices
- Partner choose α
  - You α = 0, β = -1
- Partner choose β
  - You α = 3, β = 1
- α always best choice

Strategy Domination

- We say that a strategy (α) strictly dominates a strategy (β) if my payoff from alpha is strictly greater than that of beta no matter what others do
- Do not play a strictly dominated strategy
- The strategy that dominates it is better in every possible case

Strategy Domination

~~X is the Best Choice~~

If X always worse than Y
Then X is dominated by Y

Prisoner’s Dilemma

- Two prisoners in separate cells
- Guard asks each to tell on the other
- If neither tell on each other
  - Both go to jail for a year
- If both tell on each other
  - Both go to jail for two years
- If one tells and the other doesn’t
  - The teller goes free, other goes to jail for 5 years

Prisoner’s Dilemma Payoffs

Prisoner 2

Prisoner 1

	Tell	Silent
Tell	-2, -2	0, -5
Silent	-5, 0	-1, -1

Payoff Matrix

- Prisoner 2 Tell
  - P1: T = -2, S = -5
- Prisoner 2 Silent
  - P1: T = 0, S = -1
- Ratting out the other prisoner strictly dominates not ratting them out

Split or Steal Game Show

- Two contestants choose secretly and simultaneously reveal split or steal
- Both split: each get half
- Both steal: both get 0
- Split/Steal:
    - Stealer gets it all
- Prisoner’s Dilemma

Split or Steal Payoffs

Player 2

Player 1

	Split	Steal
Split	½, ½	0, 1
Steal	1, 0	0, 0

Payoff Matrix

- Stealing weakly dominates Splitting
- Game creates drama by allowing players to talk to each other and try and make a deal

https://youtu.be/S0qjK3TWZE8

Important Note

- Decision making using game theory assumes the payoff matrix is accurate for your utility
- Example 1: $1,000,000 might not have 2x the value to you as $500,000
- Example 2: You might not care as much about ‘beating’ the opponent or getting higher payoff
- Direct translation from real-life prizes into payoff matrix may be difficult

Grades Game Payoffs

Partner

Me

	ɑ	B
ɑ	B-, B-	A, C
β	C, A	B+, B+

Grade Results

- These are the grades obtained by both players in grade game
- What might a sample payoff matrix look like for different types of players?

Different Payoff

- Previous Payoff (Greedy Players)
  - If both get B-, payoff is neural (0)
  - If I get A and you get C, payoff is good (3)
- Example New Payoff (Caring Players)
  - (A, C) was (3), but now becomes (-1)
    - “I care that the other person got less”
  - (C, A) was (-1), now becomes (-3)
    - “I am angry that they made me get less”

Grades Game Payoffs

Partner

Me

	α	β
α	0, 0	3, -1
β	-1, 3	1, 1

Payoff Matrix

- In this example, both players are greedy and assign higher payoff to their own high grade
- They don’t factor in what happens to the other player at all

Grades Game – Diff Payoffs

Partner

		α	β
Me	α	0, 0	-1, -3
	β	-3, -1	1, 1

Payoff Matrix

- What if players cared about each other?
  - -1 = 3-4 (guilt)
  - -3 = -1-2 (anger)
- No dominated strategy
- Best case: β gets 1
- Worst case: α gets -1

Grade Game – Mixed Players

- What if we are greedy, and we know that the other player is caring?
- Will choice be different if we know the strategy / payoffs of the other player are different?

Grades Game – Greedy vs Caring

Partner

		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Greedy Players

Partner

		α	β
Me	α	0, 0	-1, -3
	β	-3, -1	1, 1

Caring Players

Grades Game – I am Greedy

		Partner	
		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Greedy Players

		Partner	
		α	β
Me	α	0, 0	-1, -3
	β	-3, -1	1, 1

Caring Players

Grades Game – Greedy Player

Caring

		α	β
Greedy	α	0, 0	3, -3
	β	-1, -1	1, 1

Payoff Matrix

- As greedy player
- Is there a dominated strategy in this game?
- α dominates β

Grades Game – I am Caring

Partner

		α	β
Me	α	0, 0	-1, -3
	β	-3, -1	1, 1

Caring Players

Partner

		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Greedy Players

Grades Game – Caring Player

Greedy

Caring	α	β
α	0, 0	-1, -1
β	-3, 3	1, 1

Payoff Matrix

- As caring player, which strategy to choose?
- Nothing dominates other
- But greedy α dominates β
- We know greedy will choose α
- Our best response to greedy selection of α is α

- Putting yourself in other’s shoes to figure out what they will do

Best Response

- The strategy which performs best against a given other strategy is a best response
- In previous example, we knew that the other player had a best choice, so we choose our best response to it
- “To know your Enemy, you must become your Enemy” – Sun Tzu

What makes a Game?

• Players	i, j	you	
• Strategy	s_i	particular strategy of i	13
	S_i	set of all strategies of i	[1..100]
	s	strategy profile	sheets
	s-i	choices for all but i	
• Payoff	$u_i(s)$	utility / payoff	$5 / 0

- Assume that these are all known values

Example Game

Player 2

Player 1

	L	C	R
T	5, -1	11, 3	0, 0
B	6, 4	0, 2	2, 0

Payoff Matrix

- Players = 1, 2
- Strategy Sets
    - S1 = [T, B]
    - S2 = [L, C, R]
- Payoffs
    - U1(T, C) = 11
    - U2(B, L) = 4

Example Game

Player 2

Player 1

	L	C	R
T	5, -1	11, 3	0, 0
B	6, 4	0, 2	2, 0

Payoff Matrix

- Does player 1 have a dominated strategy?
- No, it doesn't
- Does player 2 have a dominated strategy?
- Yes, C dominates R

Example Game

Player 2

Player 1

	L	C	R
T	5, -1	11, 3	0, 0
B	6, 4	0, 2	2, 0

Payoff Matrix

- Does player 1 have a dominated strategy?
- No, it doesn't
- Does player 2 have a dominated strategy?
- Yes, C dominates R

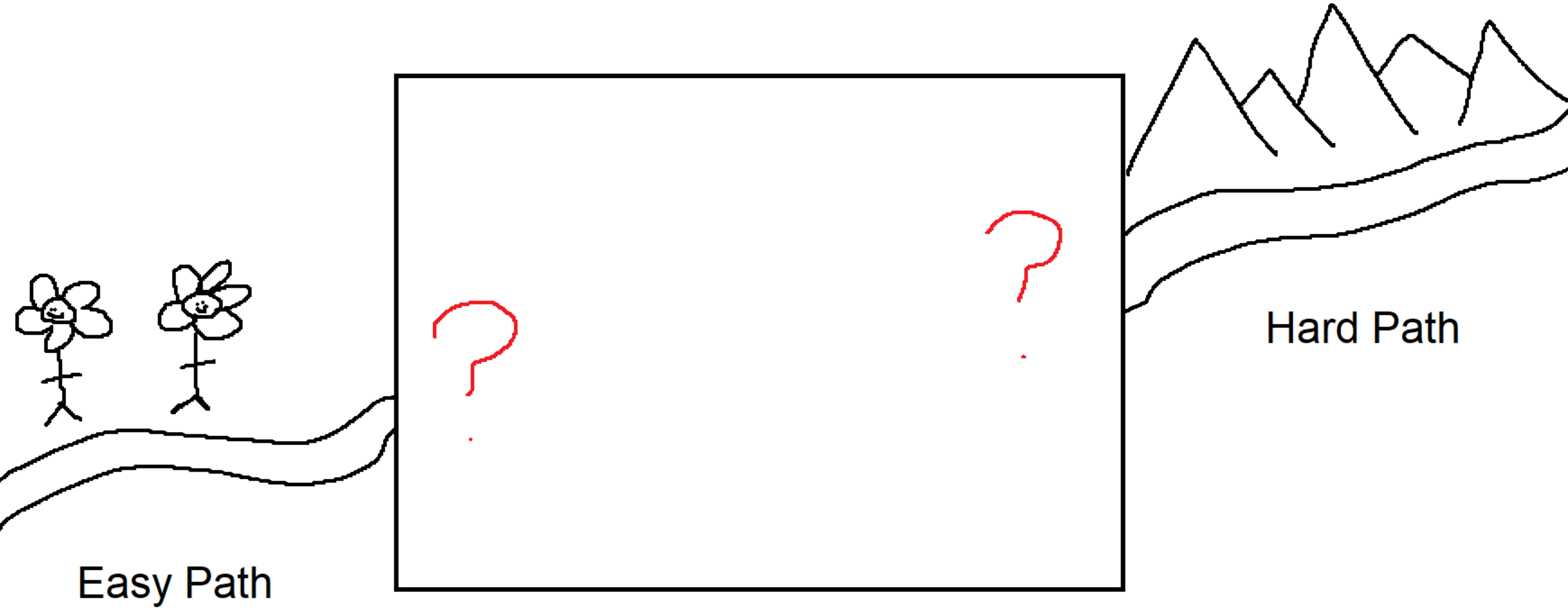
Strictly Dominated Strategy

- Player i’s strategy s_i’ is strictly dominated by player i’s strategy s_i if:

- $u_i(s_i, s\text{-}i) > u_i(s_i', s\text{-}i)$ for all s-i

Game: Defense Choice

- You are a general defending a city
  - You have an army, attacker has 2 armies
- There are two paths leading into the city
  - “Easy” path, “Hard” Path
- You can only defend one of these paths
- If attacker chooses hard path, they lose 1
- If your armies meet, attacker loses 1 army
- Enemy wants to maximize armies in city


?
?
Hard Path
Easy Path

Defense Game

Attacker

		E	H
Defense	E	1, 1	1, 1
	H	0, 2	2, 0

Payoff Matrix

- Attacker Payoff
  - Armies that reach city
- Defense Payoff
  - Attacking armies killed
- Does defense have a dominated strategy?

Defense Game

Attacker

		E	H
Defense	E	1, 1	1, 1
	H	0, 2	2, 0

Payoff Matrix

- Does attacker have a dominated strategy?
- Easy weakly dominates hard, in that it will do at least as well or better than hard

Weakly Dominated Strategy

- Player i's strategy s_i' is weakly dominated by player i's strategy s_i if:

- $u_i(s_i, s\text{-}i) >= u_i(s_i', s\text{-}i)$ for all s-i
- $u_i(s_i, s\text{-}i) > \quad u_i(s_i', s\text{-}i)$ for at least one s-i

Defense Game

Attacker

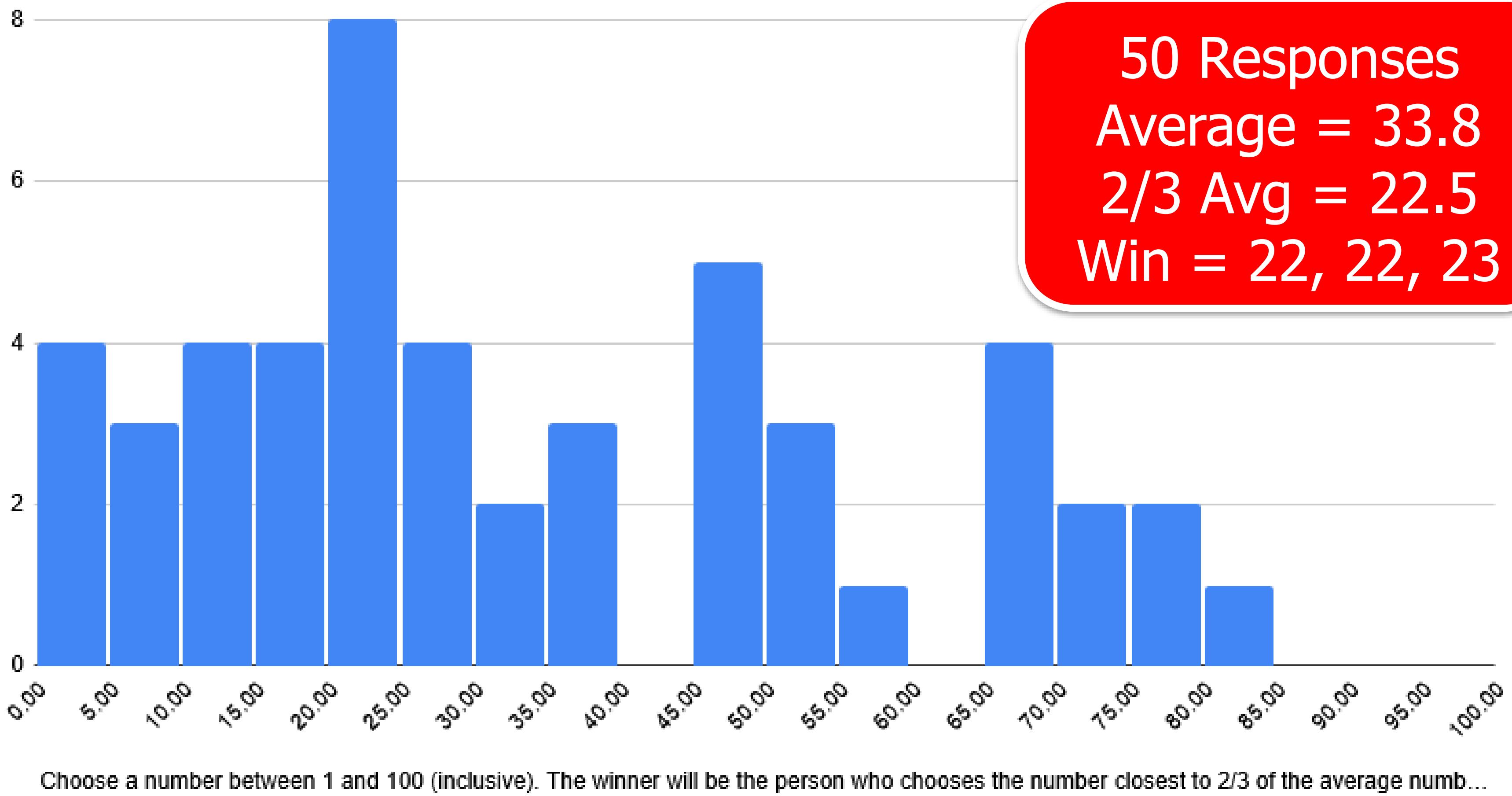
		E	H
Defense	E	1, 1	1, 1
	H	0, 2	2, 0

Payoff Matrix

- You probably shouldn’t play a weakly dominated strategy
- If you know attacker will take easy road, then defend easy

Recap

- Games have players, strategies, payoffs
- Players want to maximize their payoff
- A strategy dominates another if it is always better (strict) or as good (weak) to choose it
- Never choose to play a dominated strategy
- If there is no overall best strategy to play, consider the other player(s) best choice first
- If they have a best choice, pick a best response



Histogram of Choose a number between 1 and 100 (inclusive). The winner will be the person who chooses the number closest to 2/3 of the average number chosen.

50 Responses
Average = 33.8
2/3 Avg = 22.5
Win = 22, 22, 23

Numbers Game

- Are there dominated strategies?

- If everyone chose max value of 100
  - Average = 100
  - 2/3 Average = 67

- [68… 100] dominated by 67

Numbers Game

- Can we keep going?
- If nobody plays a dominated strategy
- Strategy set is now [1… 67]
- Everyone picks max of 67
  - Average is 67
  - 2/3 of average is 45
- So once we delete [68… 100], [46… 67] become dominated strategies (dominated by 45)
- Keep repeating… down to 1

Rationality Chain

- [68…. 100] Rational
- [46… 67] Rational + Know Others Are
- [31… 45] Rational + … + …
- …
- Common Knowledge
    - I know that you know that I know… (to inf)

Best Response

		Greedy	
		α	β
Caring	α	0, 0	-1, -1
	β	-3, 3	1, 1

Payoff Matrix

- As caring player, which strategy to choose?
- Nothing dominates other
- But greedy α dominates β
- We know greedy will choose α
- Our best response to greedy selection of α is α
- “The best you can do, given your belief about what other people are doing”

Nash Equilibrium (NE)

- A strategy profile (s_1*, s_2*, …, s_n*) is a Nash Equilibrium if for each player i, their choice is a best response to the other players’ choices

Motivations for NE

1. No Regrets
   - If we hold the strategies of everyone else fixed, no individual player i has any strict incentive to deviate (change their strategy)
2. Self Fulfilling
   - If everyone believes everyone else is going to play the NE, then they will

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

Nash Equilibrium Example

Player 2

		L	C	R
Player 1	U	0, 4	4, 0	5, 3
	M	4, 0	0, 4	5, 3
	D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$?

Nash Equilibrium Example

Player 2

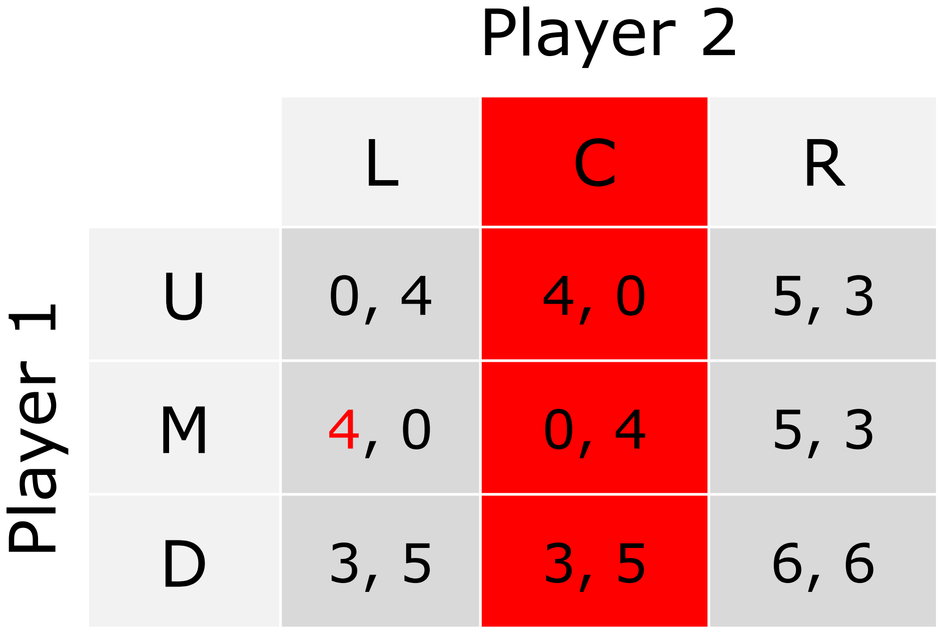
Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M

Nash Equilibrium Example



Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$?

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$?

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$? D

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$? D

- What is $BR_2(U)$?

Nash Equilibrium Example

Player 2

		L	C	R
Player 1	U	0, 4	4, 0	5, 3
	M	4, 0	0, 4	5, 3
	D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$? D

- What is $BR_2(U)$? L

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

• What is $BR_1(L)$?	M
• What is $BR_1(C)$?	U
• What is $BR_1(R)$?	D
• What is $BR_2(U)$?	L
• What is $BR_2(M)$?	

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$? D

- What is $BR_2(U)$? L
- What is $BR_2(M)$? C

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

• What is $BR_1(L)$?	M
• What is $BR_1(C)$?	U
• What is $BR_1(R)$?	D
• What is $BR_2(U)$?	L
• What is $BR_2(M)$?	C
• What is $BR_2(D)$?	

Nash Equilibrium Example

Player 2

		L	C	R
Player 1	U	0, 4	4, 0	5, 3
	M	4, 0	0, 4	5, 3
	D	3, 5	3, 5	6, 6

Payoff Matrix

What is $BR_1(L)$?	M
What is $BR_1(C)$?	U
What is $BR_1(R)$?	D
What is $BR_2(U)$?	L
What is $BR_2(M)$?	C
What is $BR_2(D)$?	R

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$? D

- What is $BR_2(U)$? L
- What is $BR_2(M)$? C
- What is $BR_2(D)$? R

- What is the NE?

Nash Equilibrium Example

Player 2

Player 1

	L	C	R
U	0, 4	4, 0	5, 3
M	4, 0	0, 4	5, 3
D	3, 5	3, 5	6, 6

Payoff Matrix

- What is $BR_1(L)$? M
- What is $BR_1(C)$? U
- What is $BR_1(R)$? D

- What is $BR_2(U)$? L
- What is $BR_2(M)$? C
- What is $BR_2(D)$? R

- What is the NE? (D,R)

Nash Equilibrium Example 2

Player 2

		L	C	R
Player 1	U	0, 2	2, 3	4, 3
	M	11, 1	3, 2	0, 0
	D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

Player 2

Player 1

	L	C	R
U	0, 2	2, 3	4, 3
M	11, 1	3, 2	0, 0
D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

Player 2

Player 1

	L	C	R
U	0, 2	2, 3	4, 3
M	11, 1	3, 2	0, 0
D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

Player 2

		L	C	R
Player 1	U	0, 2	2, 3	4, 3
	M	11, 1	3, 2	0, 0
	D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

		Player 2		
		L	C	R
Player 1	U	0, 2	2, 3	4, 3
	M	11, 1	3, 2	0, 0
	D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

Player 2

		L	C	R
Player 1	U	0, 2	2, 3	4, 3
	M	11, 1	3, 2	0, 0
	D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

Player 2

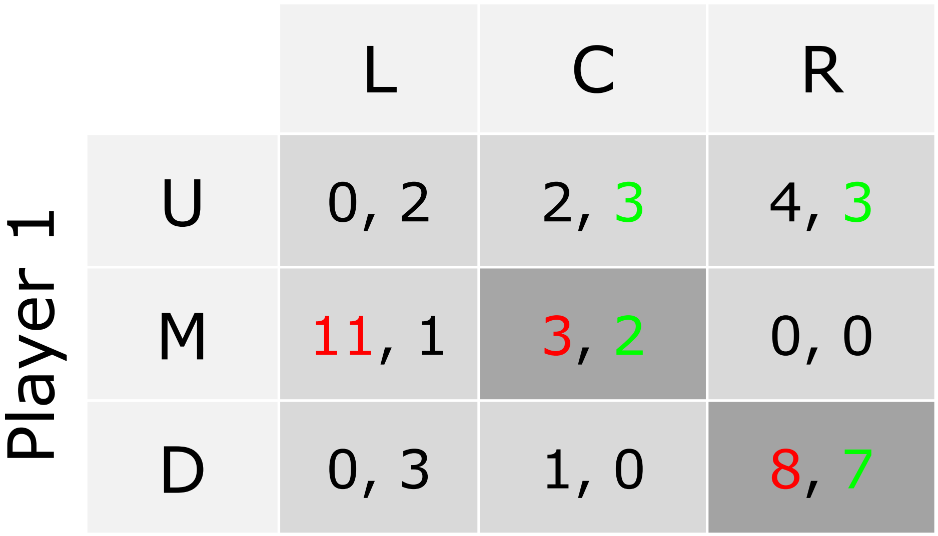
Player 1

	L	C	R
U	0, 2	2, 3	4, 3
M	11, 1	3, 2	0, 0
D	0, 3	1, 0	8, 7

Payoff Matrix

Nash Equilibrium Example 2

Player 2



Player 1

	L	C	R
U	0, 2	2, 3	4, 3
M	11, 1	3, 2	0, 0
D	0, 3	1, 0	8, 7

Payoff Matrix

Relate NE to Dominance

Partner

		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Payoff Matrix

- α strictly dominates β
- Let’s find the NE

Relate NE to Dominance

Partner

Me

	α	β
α	0, 0	3, -1
β	-1, 3	1, 1

Payoff Matrix

- α strictly dominates β
- Let’s find the NE

Relate NE to Dominance

Partner

		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Payoff Matrix

- α strictly dominates β
- Let’s find the NE

Relate NE to Dominance

Partner

		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Payoff Matrix

- α strictly dominates β
- Let’s find the NE

Relate NE to Dominance

		Partner	
		α	β
Me	α	0, 0	3, -1
	β	-1, 3	1, 1

Payoff Matrix

- α strictly dominates β
- Let’s find the NE

Relate NE to Dominance

Partner

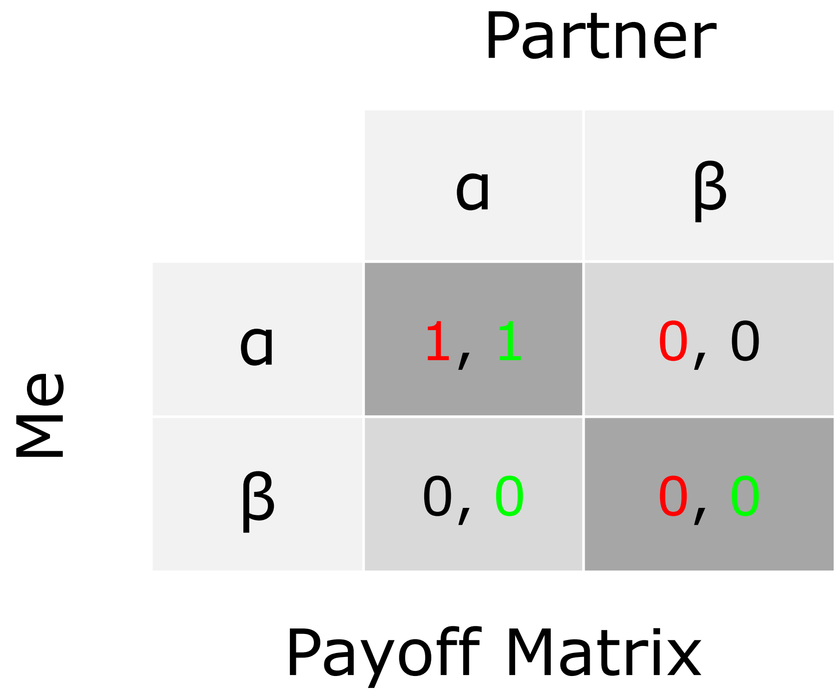
Me

	α	β
α	0, 0	3, -1
β	-1, 3	1, 1

Payoff Matrix

- α strictly dominates β
- Let’s find the NE
- No strictly dominated strategy can ever be played in an NE

Relate NE to Dominance



Partner

Me

	α	β
α	1, 1	0, 0
β	0, 0	0, 0

Payoff Matrix

- More than one NE can exist in a game
- Weakly dominated strategies can be chosen
- In a NE, no player can do STRICTLY better by changing their strategy

Investment Game

- Players: Everyone
- Strategy Set: [Invest $100, Don’t Invest]
- Payoffs:
  - If you don’t invest, you gain/lost nothing
  - If you invest
    - If >= 80% of people invested, you gain $100
    - If < 80% of people invested, you lost $100

Investment Game NE

- What is the NE for Investment Game?

- Two Nash Equilibria:
  - Nobody Invests
  - Everyone Invests

Possible Exam Questions

- Given a payoff matrix, compute the dominated / dominating strategies
- Explain why a given choice of strategy is good or bad for a given matrix
- Compute the best response to a strategy
- Construct matrix payoffs such that a given strategy dominates another

Possible Exam Questions

- Definition of Nash Equilibrium
    - Mathematical
    - English Description
- Calculate NE for given matrix
    - None / One / Multiple
- Explain NE for a described game